Limits and Continuity of Function

Among other things, calculus looks at what happens to a function as the independent variable approaches a particular value.

In some cases, the number can be substituted to find the limiting value, *e.g.*, if **f(x) = 4x -7**, what is the limit as x approaches 5? Lim_ (x→ 5) f(x) = 4(5) – 7 = 13

However, if **g(x) = (x^2 – 2x - 3)/(x – 3)** and you have to find the value of this function as x approaches 3, you cannot substitute 3 in this expression because the denominator will become 0 and it will be undefined.
In this example, we can factor the numerator:

$x^2 - 2x - 3 = x^2 - 3x + x - 3 = x (x - 3)+1 (x - 3) = (x - 3) (x + 1)$

So $\lim_{x \to 3} g(x) = (x -3) (x + 1)/(x - 3) = x + 1 = 4$

The other way to find the limiting value of g(x) is to find the left hand and right hand limits of g(x).

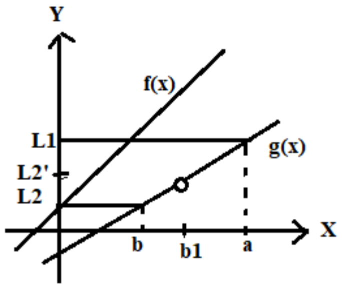
Left hand limit: (approaching 3 from below 3)

x	2.7	2.75	2.85	2.9	2.95
g(x)	3.7	3.75	3.85	3.9	3.95

Right hand limit: (approaching 3 from above 3)

x	3.05	3.2	3.3	3.4	3.5
g(x)	4.05	4.2	4.3	4.4	4.5

Note here also, that as g(x) approaches or “tends to” 3, g(x) tends to 4.
Again, refer to the tutorial for the concept of how left hand and right hand limits are different when the function is discontinuous.



A

In graph **A**, let us find the **limit** of **f(x)** as **x** approaches or tends to **b**. **f(x)** is a continuous line.

lim f(x) = ?
x→b

The **left hand limit** of **f(x)** as **x** tends to **b** is **L1**. And the **right hand limit** of **f(x)** as **x** tends to **b** is also **L1**. Thus, the **limit** of **f(x)** as **x** approaches **b** is **L1**.
It is the same as evaluating **f(x)** at **x equals b**, that is, **f(b).**

lim f(x) = L1; lim f(x) = L1 = f(b)
x→ b- x→b+

What is the **limit** of **g(x)** as **x** tends to **b1**?

lim g(x) = ?
x→b1
In graph **A**, note that **g(x)** has an open circle at (**b1, L2')**.
This means that **g(x)** does not exist at this point.
Let us find the **limit** of **g(x)** as **x** approaches **b1**.
The **left hand** and **right hand limits** are **L2'** as **x** approaches **b1**.

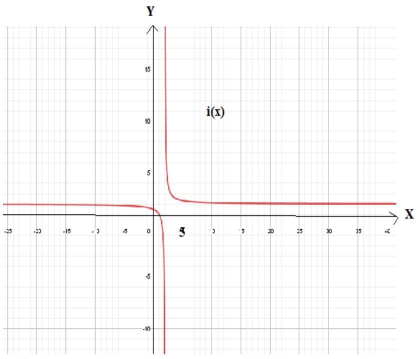
$$\lim_{x \to b1-} g(x) = \lim_{x \to b1+} g(x) = L2'$$

But **g(x)** itself does not exist at **x = b1**.

But g(b1) Does Not Exist (DNE)

However, **g(x)** can be evaluated at **x = b** and **x = a**.
And these values are the same as the **limits** of **g(x)** as **x** approaches **b** and **a**.

$$\lim_{x \to b} g(x) = g(b) = L2; \quad \lim_{x \to a} g(x) = g(a) = L1$$



C

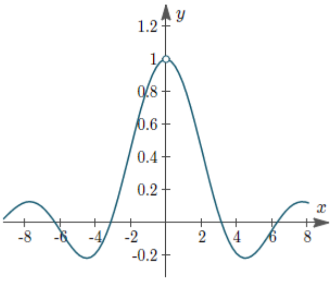
In graph **C**, **i(x)** has two parts.
The first part is the upper right one.
Both arms extend towards **infinity** (∞).

The second part is the lower left one.
Both arms extend towards **negative infinity** (-∞).

$$\lim_{x \to \infty} i(x) = ? \quad \lim_{x \to -\infty} i(x) = ?$$

$$\lim_{x \to \infty} i(x) = 2; \quad \lim_{x \to -\infty} i(x) = 1$$

Suppose **h(x) = sin x/x**; what is lim h(x) = ? Here too, you cannot substitute 0 because it is
x → 0 undefined. Graph this function, either manually by finding sin x for different x values between -10 and 10 OR by entering the function in GeoGebra.



Graph of $y = \frac{\sin(x)}{x}$.

This is the graph and you see that as x approaches 0, limit of sin x/x is 1. However, at x = 0, the function itself is undefined and does not exist so there is an open circle/hole.

Check out the tutorial for the example of limit as x approaches infinity.

What if the variable x is in the denominator and you want to know the limit as x ✉ ∞?

lim **i(x) = (5 – 3x)/(6x + 1)** = ?
x→ ∞

Remember that if x = ∞, 1/x = 0

So lim i(x) = (5/x – 3x/x)/(6x/x + 1/x) = (5 (0) – 3)/(6 + 0) = -3/6 = -1/2
x→ ∞

<u>Continuous and discontinuous functions</u>:

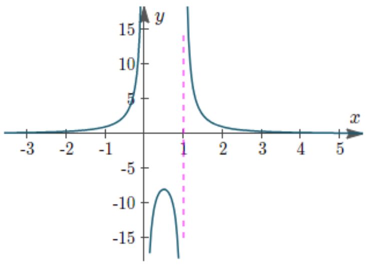
Criteria for continuous functions

Note that for a **function f(x)** to be called **continuous**,

- **f(a)** should be defined
- lim_(x→ a) f(x) exists
- lim_(x→ a) f(x) = f(a)

When any value of x gives you a corresponding value of y and there are no “gaps” or “breaks” in the curve, the function is said to be continuous.

As seen in the example of i(x) in the tutorial, for function **y = 2/(x^2 – x)**, a small change in x near x = 0 and 1, will give you a very large change in the value of y, besides there being gaps in the function.



The Graph of $f(x) = \frac{2}{x^2 - x}$, a discontinuous function. function is not defined for x = 0 and x = 1. Here, the curve gets close to but doesn’t touch the x and y axes and the dotted line at x = 1. These are three **asymptotes** for this function.