Integration using GeoGebra

Integral calculus is useful for finding areas, volumes, central points and the direct relationship between two variables, unlike differentiation which deals with the rate of change of two variables.

Let's say we are comparing small changes in **x** and **y**, with **y** being some function **f** of **x**.

Differential **dy = f'(x) dx**

Example 1: Thus, if $\mathbf{y = 3x^5 - x}$, the differential $\mathbf{dy = f'(x)\,dx = (15x^4 - 1)\,dx}$

Integration is the opposite of differentiation.

So in Example 1, $\mathbf{dy/dx = 15x^4 - 1}$

To find the integral of $\mathbf{15x^4 - 1}$, we must ask what do we differentiate to get $\mathbf{15x^4 - 1}$?

Integral of a Power of x and Integral of a Constant

$\mathbf{15 \int x^4\,dx - \int 1\,dx = 15\,(x^{4+1}/4+1) - x + C = 15\,(x^5/5) - x + C = 3x^5 - x + C}$

As **C** can take different values, $\mathbf{3x^5 - x + C}$ represents a family of integrals, equivalent to the **indefinite integral**.

As mentioned, integration helps you to find the **area under a curve (AUC)**. Alternatively, you can divide the space under the curve into shapes such as rectangles or trapezoids or parabolas or cubics, and add the areas of these shapes to get the **AUC**.

A German mathematician, **Bernhard Riemann**, came up with this idea. As these shapes obviously will not fill up the AUC exactly, the **Riemann sum** is only an **approximation** of the AUC. As these shapes get smaller, the sum of their areas will approach the Riemann integral.

There is a lot of information available on left, right, upper, lower Riemann sums, the midpoint rule and trapezoidal rule.

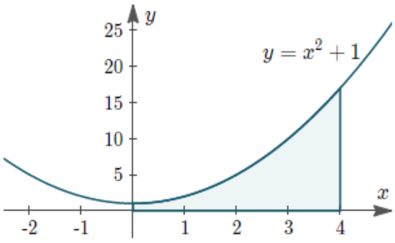
To avoid all this work of adding up rectangles, we can use integration, specifically, the **definite integral** as the area under a function **y = f(x)** from **x = a to x = b** as $\int_a^b f(x)\,dx$.

The definite integral can be calculated by assuming that if **F(x)** is the integral of **f(x)**, then

$$\boxed{\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)}$$

Let's look at **examples 2-5** to see how integration can be useful to us and at **examples 6-7** to see how to calculate the integral using **numerical methods**.

Example 2: If we had to find AUC $\mathbf{y = x^2 + 1}$ between $\mathbf{x = 0, x = 4}$ and $\mathbf{y = 0,}$



Area bounded by a parabola, the axes and $x = 4$..

$\int_0^4 (x^2 + 1)\, dx$ = $\left[\frac{x^3}{3} + x\right]_0^4$ = $(4^3/3 + 4) - (0^3/3 + 0) = 76/3$ units

Example 3: Find the work done if a force **F(x) = sqrt (2x – 1)** is acting on an object and moves it from **x = 1** to **x = 5**.

$\int_1^5 (2x - 1)^{1/2}\, dx$ = $\frac{1}{3}\left[(2x-1)^{3/2}\right]_1^5$ = $\frac{1}{3}\left[9^{3/2} - 1^{3/2}\right]$ = 1/3 (27 – 1) = 26/3

Example 4: Find the average value of $y = x\,(3x^2 - 1)^3$ from 0 to 1.

$$\int_0^1 x\left(3x^2-1\right)^3 dx = \frac{1}{24}\left[\left(3x^2-1\right)^4\right]_0^1 = \frac{1}{24}\left[\left(3(1)^2-1\right)^4 - \left(3(0)^2-1\right)^4\right]$$

= 1/24 [16 – 1] = 15/24 = 5/8

Example 5: Find the displacement of an object from **t = 2** to **t = 3**, if the velocity of the object at time **t** is given by $\mathbf{v = (t^2 + 1)/(t^3 + 3t)^2}$.

Suppose $\mathbf{u = t^3 + 3t, du = (3t^2 + 3)\, dt = 3\,(t^2 + 1)\, dt}$

$\mathbf{du/3 = (t^2 + 1)\, dt}$

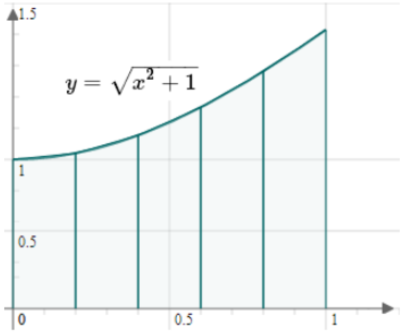
$$\int_2^3 \frac{t^2+1}{\left(t^3+3t\right)^2} = \frac{1}{3}\int_{t=2}^{t=3} \frac{1}{u^2}\,du = -\frac{1}{3}\left[\frac{1}{u}\right]_{t=2}^{t=3} = -\frac{1}{3}\left[\frac{1}{t^3+3t}\right]_2^3$$

$$-\frac{1}{3}\left[\frac{1}{36} - \frac{1}{14}\right] = 0.014550$$

Example 6: Find $\int_0^1 (x^2 + 1)^{1/2}\, dx$

If you let $\mathbf{u = x^2+1}$, then you need to find **du = 2x dx**. But there is only **dx** not **2x dx** or **2x** so **du** cannot be used here correctly. Let’s see how to use numerical methods.

Let's look at the **trapezoidal rule** for **Example 6**. We will draw 5 trapezoids under the curve.



a = 0, **b** = 1, width of each trapezoid **is Δx = (b – a)/n** = 0.2

y_0 = f(a) = f(0) = sqrt (0^2 + 1) = 1

y_1 = f(a + Δx) = f (0.2) = sqrt (0.2^2 + 1) = 1.0198039

y_2 = f(a + 2Δx) = f (0.4) = sqrt (0.4^2 + 1) = 1.0770330

y_3 = f(a + 3Δx) = f (0.6) = sqrt (0.6^2 + 1) = 1.1661904

y_4 = f(a + 4Δx) = f (0.8) = sqrt (0.8^2 + 1) = 1.2806248

y_5 = f(a + 5Δx) = f (1.0) = sqrt (1^2 + 1) = 1.4142136

Trapezoidal rule states that,

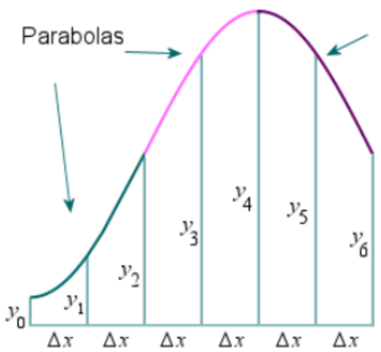
$$\text{Area} = \int_a^b f(x)\,dx \approx \Delta x\left(\frac{y_0}{2} + y_1 + \ldots + \frac{y_n}{2}\right)$$

Integral ~ 0.2 (1/2 x 1 + 1.0198039 + 1.0770330 + 1.1661904 + 1.2806248 + ½ x 1.4142136)

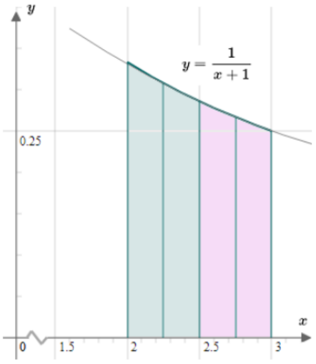
= 1.150

Compare this to your results in GeoGebra.

Simpson' rule uses parabolas to approximate each part of the curve. It is even more accurate than the trapezoidal rule and the Riemann sums.



Example 7:



Approximate $\int_2^3 \frac{dx}{x+1}$ using Simpson's rule with n = 4 (n has to be even in Simpson's rule).

Simpson's rule states:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}[\text{FIRST} + 4(\text{sum of ODDs}) \ +2(\text{sum of EVENs}) + \text{LAST}]$$

So here, $\Delta x = (b-a)/n = (3 – 2)/4 = 0.25$

$y_0 = f(a) = f(2) = 1/(2+1) = 0.3333333$

$y_1 = f(a + \Delta x) = f (2.25) = 1/(2.25 + 1) = 0.3076923$

$y_2 = f(a + 2\Delta x) = f (2.5) = 1/(2.5 + 1) = 0.2857142$

$y_3 = f(a + 3\Delta x) = f (2.75) = 1/(2.75 + 1) = 0.2666667$

$y_4 = f(b) = f (3) = 1/(3 + 1) = 0.25$

$\int_2^3 \frac{dx}{x+1}$ ~ 0.25/3 (0.333333 + 4 (0.3076923 + 0.2666667) + 2 (0.2857142) + 0.25)

= 0.2876831

Some 2000 years before **Newton** and **Leibniz** developed differential calculus in the 17th century, **Archimedes** had worked out good approximations of the area of a circle and the value of **pi** (π).

Go ahead and draw a parabola $\mathbf{y = x^2}$ in GeoGebra and draw a line **AB**, such that A is (-1, 1) and B is (2, 4). But remember that Archimedes did not use the Cartesian co-ordinate system as it was invented by **Descartes** in the 17th century.

He constructed a triangle **ABC** where **x(C)** is half-way between **x(A)** and **x(B)** and **C** is on the parabola. Archimedes showed that the area of the parabolic segment (area of parabola below line **AB**) is 4/3 of the area of triangle **ABC**. He used the **Method of Exhaustion** and used smaller polygons to fill the shape and then added the areas of these polygons to find the area of the curved shape.

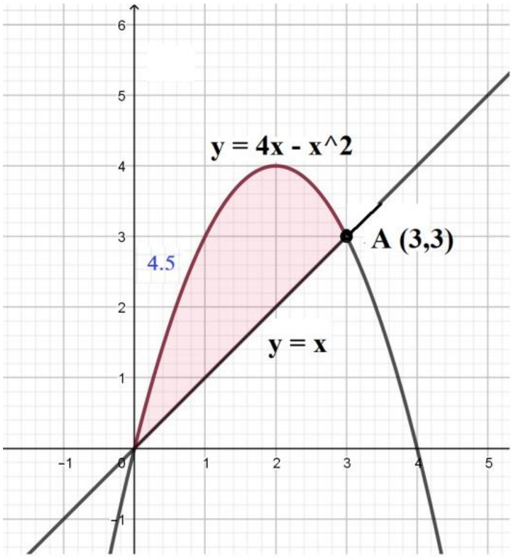
You can choose **D** on the parabola so that **x(D)** is half-way between **x(A)** and **x(C)**. Keep going on to draw **E** on the parabola so that **x(E)** is half-way between **x(C)** and **x(B)**. Now add the areas of triangles **ABC**, **ACD** and **BCE**. Use GeoGebra to calculate the area of the parabolic segment. Compare this to the sum of the areas of the three triangles. Are they approximately equal? You can even continue to draw triangles to fill the rest of the segment.

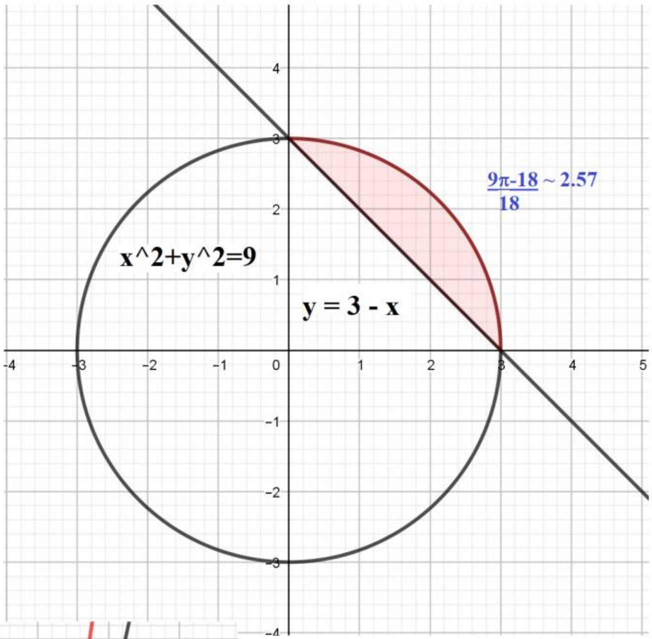
If we used integration, we would go for the area between 2 functions, the parabola **($y_1 = x^2$)** and the line **($y_2 = x + 2$)**.

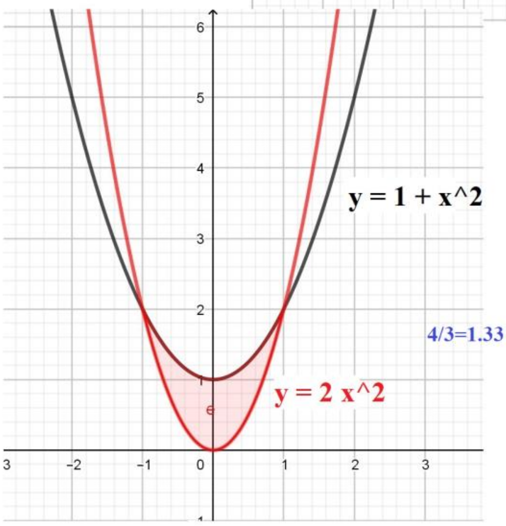
$$\int_a^b (y_2 - y_1)dx = \int_{-1}^{2} [(x+2) - x^2]dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = 4.5$$

Fun facts: Integration was used to design the Petronas Towers in Kuala Lumpur for strength to withstand high forces due to winds.

Historically, integration was used to find the volumes of wine casks, which have curved surfaces.







Reference: https://www.intmath.com/